Int. J. Solids Structures, 1969, Vol. 5, pp. 289 to 302. Pergamon Press. Printed in Great Britain

LIMIT ANALYSIS OF CYLINDRICAL SHELLS BY DYNAMIC PROGRAMMING

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Abstract—Determining the smallest upper bound on the collapse load of a rotationally symmetric shell can be reduced to the problem of minimizing an integral subject to certain initial conditions. Dynamic programming gives a way of carrying out this minimization which is not limited by lack of smoothness in the integrand function, and leads to a convenient computational algorithm. The paper develops the method and applies it to a thin cylindrical shell under ring loading; the results agree closely with exact analysis.

INTRODUCTION

THE solutions to large classes of problems in solid mechanics are known to satisfy certain minimum principles. In elasticity, for example, the minimum complementary energy and minimum potential energy theorems have been known for a long time, and have been used extensively, both to find exact solutions and to obtain bounds on forces and deflections. Corresponding principles in plasticity and creep are well known [1–3], but except for the limit theorems of plasticity [4] they have been relatively little used. Minimum principles in plasticity and their numerical application have recently been reviewed by Hodge [5]. The formulation of a physical problem as a minimization problem is often reasonably straightforward, but the solution of the resulting mathematical problem may be difficult. Classical methods based on the calculus of variations are rarely appropriate, since they require severe restrictions on continuity and differentiability; as the exact solutions of simple problems lead us to expect, stress and velocity discontinuities will often occur.

A number of mathematical techniques have been applied to the solution of minimum problems in plasticity, but the field has been little explored. Linear programming has been extensively applied to frame analysis (see, for example [6]) and recently extended to plate problems [7]. Hodge and Biron [8–10] have applied the SUMT technique to shell analysis and torsion. In this paper a different technique, dynamic programming, is applied to a shell analysis problem. A separate paper [11] discusses its application to plastic design.

The theory of dynamic programming has been developed by Bellman *et al.*, who have described it in a number of books [12–15]. The theory is widely known among economists and statisticians, but except in control theory it has been little used by engineers. It is concerned with processes in which a number of decisions have to be made and the effects of different decisions interact. Two characteristic ideas recur all its applications, the idea of sequential decisions (decisions made one at a time instead of all at once), and imbedding (a single problem is imbedded in a wider family of related problems). A sequence of decisions is called a policy. Simple arguments lead to the subtle though intuitively clear principle of optimality:

An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with respect to the state resulting from the first decision [12].

The method is attractive for plasticity problems because it requires only weak restrictions on smoothness and continuity of functions involved, and because it is not restricted by non-linearity or by constraints. In the problem considered later in the paper it is applied to integral minimization and a short outline of the development of the technique to be used is given in the next section of the paper; it follows Bellman [12].

Dynamic programming

Consider the following problem:

minimize $$\int_0^{T_1} F(x, y)\,\mathrm{d}t \tag{1}$$

subject to $$\frac{\mathrm{d}x}{\mathrm{d}t} = g(x, y) \tag{2}$$

$$x(0) = c_1. \tag{3}$$

One interpretation is this: x is a variable defining the state of a dynamic system at time t. The current value of a control variable y determines the rate at which x changes; c_1 is the initial value of x. The problem is to choose y so as to minimize a function of x and y integrated over time T_1. Using the idea of imbedding, the parameters c and T may take any values, not necessarily those of the original problem; to indicate this they are now written without subscripts. Whatever the minimum value of the integral is, it can be a function only of c and T; denote the minimum value by $f(c, T)$. If $y^*(t)$ is the function which minimizes the integral then for any $S > 0$

$$f(c, T) = \int_0^S F(x, y^*)\,\mathrm{d}t + \int_S^T F(x, y^*)\,\mathrm{d}t. \tag{4}$$

Any choice of $y(t)$ over the interval $[0, S]$ will have the effect of changing x through the differential equation (2), and will transform the initial value c into a new value of x at S; call this $c(S)$. Whatever decisions are made in $[0, S]$, there is over the remaining interval $[S, T]$ a problem of exactly the same form as the original problem, with the difference that the initial value c is now $c(S)$ and the time T is now $T-S$. If the choice of y in $[0, S]$ is the optimal choice $y^*(t)$ and transforms c into $c^*(S)$ equation (4) can be rewritten

$$f(c, T) = \int_0^S F(x, y^*)\,\mathrm{d}t + f(c^*(S), T-S). \tag{5}$$

Observe that this implies a shift in the integration range for the minimized integral corresponding to the second term on the right-hand side of (4), from $[S, T]$ to $[0, T-S]$, but that since t does not appear explicitly in the integrand this has no effect. Since $y(t)$ has to be chosen so as to yield the minimum value $f(c, T)$, equation (5) gives us the basic fundamental equation

$$f(c, T) = \min_{y[0,S]} \left[\int_0^S F(x, y)\,\mathrm{d}t + f(c(S), T-S) \right]. \tag{6}$$

If S is made small, the choice of $y(t)$ over the interval becomes a choice of $y(0)$; if we write $y(0)$ as v, then

$$f(c, T) = \min_v [F(c, v)S + f(c + g(c, v)S, T-S)] + o(S). \tag{7}$$

The argument can be continued, regarding S as an infinitesimal and locating the minimum by calculus. Not surprisingly, this yields the Euler equations and many of the classical results of the calculus of variations [12, 15]. If numerical solutions are wanted this may not be the best approach, particularly if $F(x, y)$ is not continuously differentiable or if there are inequality constraints on y. Instead (7) itself gives a computational algorithm for constructing a solution. If instead of allowing T to vary continuously we restrict it to the set of values

$$T = 0, \Delta, 2\Delta, \ldots, k\Delta, \ldots, r\Delta$$

and let $S = \Delta$, equation (7) becomes the recurrence relation

$$f(c, k\Delta) = \min_{v}[F(c, v)\Delta + f(c+g(c, v)\Delta, (k-1)\Delta)] \tag{8}$$

while by definition

$$f(c, 0) = 0 \tag{9}$$

for all c. Setting $k = 1$ equation (8) enables us to construct $f(c, \Delta)$ for any c. Setting $k = 2$, and now knowing $f(c, \Delta)$ a second use of (8) enables us to construct $f(c, 2\Delta)$ as a function of c. Repeating this process $f(c, k\Delta)$ can be constructed for any value of k, and therefore for any T.

The solution to the original problem is $f(c_1, T_1)$. If at each step the minimizing value of v has been recorded as a function of c and k, $v(c, k\Delta)$, then the minimizing function $y^*(t)$ can be constructed. Constructing $y^*(t)$ starts with the initial value c_1 of the original problem but at the *last* of the Δ-steps of the first part of the calculation where $f(c, T)$ and $v(c, T)$ were constructed. Starting at $T_1 = r\Delta$, $v(c_1, r\Delta)$ is y^* over $[0, \Delta]$, and transforms c_1 into $c_1+g(c_1, v(c_1, r\Delta))\Delta$. In the next interval $[\Delta, 2\Delta]$, y^* is

$$v(c_1+g(c_1, v(c_1, r\Delta))\Delta, (r-1)\Delta)$$

and this in turn transforms c into a new value. Repeating this process constructs $y^*(t)$ over $[0, T)$.

Exactly the same process of argument can be applied to the problem with more than one dependent variable [12]:

minimize
$$\int_0^T F(x_1, x_2, y)\,\mathrm{d}t \tag{10}$$

subject to
$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = g_1(x_1, x_2, y) \tag{11}$$

$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = g_2(x_1, x_2, y) \tag{12}$$

$$x_1(0) = c_1 \tag{13}$$

$$x_2(0) = c_2. \tag{14}$$

If the minimum (subject to the constraints) is written $f(c_1, c_2, T)$ then the basic functional equations corresponding to (8) and (9) are

$$f(c_1, c_2, T) = \min_{v}[F(c_1, c_2, v)\Delta + f(c_1+g_1(c_1, c_2, v)\Delta, c_2+g_2(c_1, c_2, v)\Delta, T-\Delta)] \tag{15}$$

$$f(c_1, c_2, 0) = 0. \tag{16}$$

In order to use these recurrence relations in computation it has to be possible somehow to store data from which the functions $f(c, k\Delta)$ can be evaluated for any c within the range of interest. How can this be done? As the simplest possibility $f(c, k\Delta)$ can be found at points on an evenly-spaced grid of values of c; intermediate values can be found by interpolation when they are required in the computation of $f(c, (k+1)\Delta)$. The design application described in [11] uses this technique. Alternatively the functions can be represented by truncated series of orthogonal polynomials, the coefficients being found by numerical integration. This method has been discussed by Bellman *et al.* [16] and is the one used here.

CYLINDRICAL SHELL UNDER CENTRAL RING LOADING

As a plasticity problem this does not have a complete closed-form analytic solution; analytic solutions completed by numerical integration have been derived by Eason and Demir [17, 18]. Two characteristics of their solutions suggested that this would be a good test of the dynamic programming method; the mode of deformation is qualitatively different for different shell lengths, and very simple velocity fields do not give good upper bounds on collapse loads. Since the solution is imbedded in a wider class of solutions it is also possible, without further computation, to compare results with Calladine's studies of edge loading on cylindrical shells [19].

An axisymmetric cylindrical shell is made from an elastic–perfectly plastic material whose yield stress in simple tension is σ_0; its length is L, its radius R and its (uniform) thickness h. It is loaded by a concentrated ring of force directed radially outward half-way along its length (Fig. 1). Axial position is described by distance x from the mid-plane.

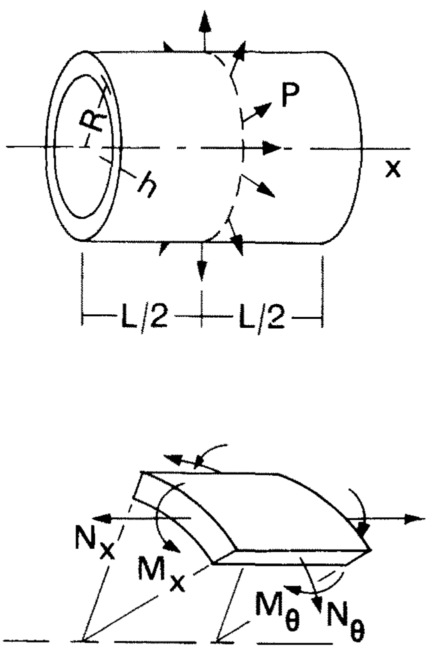

FIG. 1.

The stress resultants acting on an element of the cylinder are defined in the diagram in Fig. 1. From equilibrium the axial direct stress resultant N_x vanishes.

At any point in the shell the displacement is radial, and the corresponding velocity is denoted U. The circumferential strain rate is U/R, and the curvature rate in the x-direction is $\mathrm{d}^2U/\mathrm{d}x^2$. The rate of energy dissipation per unit area of the shell is

$$D = \frac{N_\theta U}{R} + M_x \frac{\mathrm{d}^2U}{\mathrm{d}x^2}. \tag{17}$$

The upper-bound limit theorem [4] determines an upper bound on the limit ring load at which collapse will occur. If $U(x)$ is any twice continuously differentiable radial velocity field and P^* is the collapse ring load intensity, then

$$P^*U(0) \leq \int_{-L/2}^{+L/2} D\left(U, \frac{\mathrm{d}^2U}{\mathrm{d}x^2}\right) \mathrm{d}x. \tag{18}$$

Introducing non-dimensional variables

$$\begin{aligned} \xi &= \frac{x}{\sqrt{(Rh)}} \quad & \lambda &= \frac{L}{\sqrt{(Rh)}} \quad & m &= \frac{M_x}{\frac{1}{4}\sigma_0 h^2} \quad & n &= \frac{N_\theta}{\sigma_0 h} \\ u &= \frac{U}{\sqrt{(Rh)}} \quad & p &= \frac{\sqrt{(Rh)}}{\sigma_0 h^2} P \quad & \Omega &= \frac{DR}{\sigma_0 h \sqrt{(Rh)}} \end{aligned} \tag{19}$$

and denoting differentiation with respect to ξ by superscript primes ($u' \equiv \mathrm{d}u/\mathrm{d}\xi$), (17) and (18) become

$$\Omega = nu + \tfrac{1}{4} m u'' \tag{20}$$

$$p^* \leq \int_{-\lambda/2}^{\lambda/2} \Omega(u, u'')\, \mathrm{d}\xi. \tag{21}$$

The way in which the kinematically admissible u and u'' determine the associated values of n and m depends on the yield condition for an element of the shell. Drucker [21] derived the yield condition for radially loaded cylindrical shells of material satisfying the Tresca yield condition; it is illustrated in Fig. 2, and is given by

$$|m| = 4|n|(1-|n|) \qquad \tfrac{1}{2} \leq |n| \leq 1 \tag{22}$$

$$|m| = 1 \qquad |n| < \tfrac{1}{2}. \tag{23}$$

The generalized strain rates corresponding to n and m are u and $\frac{1}{4}u''$, and [21]

$$\Omega = \begin{cases} |u| & \text{if } |u| > |u''| \\ \frac{1}{4}\left(2|u| + |u''| + \dfrac{u^2}{|u''|}\right) & \text{if } |u| \leq |u''| \end{cases} \tag{24}$$

the first expression corresponding to the corners (0, 1) and (0, −1) of the yield locus, and the second to the parabolic sides.

Since Ω is homogeneous of order one, two velocity fields $u_1(x)$ and $u_2(x)$ which only differ by a multiplying constant ($u_2(x) = au_1(x)$, a independent of x) will give the same upper bound on p. Restricting attention to velocity fields with $u = 1$ at $x = 0$ will not therefore produce any loss in generality.

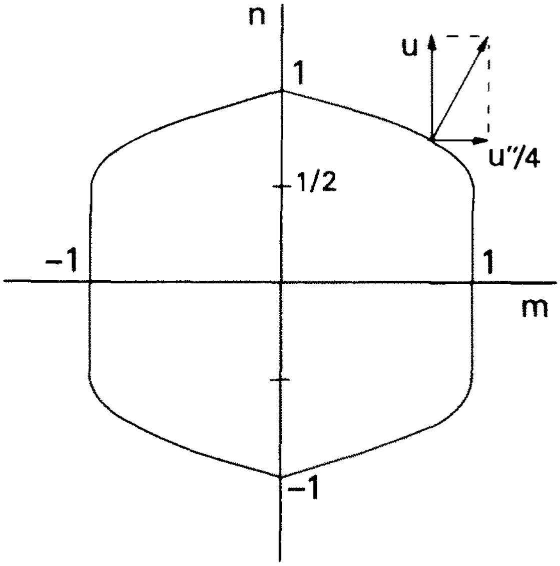

FIG. 2.

Since the cylinder is symmetric about $x = 0$ the velocity field which minimizes the upper bound can be expected itself to be symmetric. It follows that the limit ring load for a cylinder of non-dimensionalized half-length $\lambda/2$ is determined by the solution of the following problem:

minimize
$$2\int_0^{\lambda/2} \Omega(u, u'')\,d\xi \tag{25}$$

subject to
$$u(0) = 1 \tag{26}$$
$$u'(0) = 0 \tag{27}$$

where Ω is given by (24).

One might modify this formulation of the problem by extending the class of admissible velocity fields to allow hinge circles at which u is continuous but u' discontinuous. However, to do this would complicate the later numerical treatment, and it is simpler to consider only velocity fields with u' continuous but to allow u'' to become large.

A direct application of the calculus of variations will yield a displacement formulation of the original problem, in which different equations will correspond to different regions, depending on the signs of u and u'', since Ω is not continuously differentiable across $u = u''$ or $u = 0$. Solutions to these equations have to be matched across the discontinuities. The difficulties of doing this are discussed in many papers; see, for example [22].

Dynamic programming gives an alternative approach, which is likely to be more convenient for numerical solution. A comparison shows that the minimization problem (25–27) is a special case of the general problem (10–14) in which ξ is identified with t, u with x_1, u' with x_2 and u'' with y, so that $g_1 = u'$, $g_2 = u''$.

If $f(c_1, c_2, \lambda/2)$ is the minimum reached by the integral of (25) subject to the initial conditions

$$u(0) = c_1 \tag{28}$$
$$u'(0) = c_2 \tag{29}$$

then from the functional equation (15)

$$f(c_1, c_2, k\Delta) = \min_{u''}[\Omega(c_1, c_2, u'')\Delta + f(c_1 + c_2\Delta, c_2 + u''\Delta, (k-1)\Delta)] \tag{30}$$
$$f(c_1, c_2, 0) = 0.$$

If this is to form the basis of an algorithm for constructing $f(c_1, c_2, k\Delta)$, data must be stored from which $f(c_1, c_2(k-1)\Delta)$ can be calculated for any c_1 and c_2. Since $\Omega(u, u'')$ is homogeneous of order 1 in $|u|$ and $|u''|$, $f(|c_1|, |c_2|, k\Delta)$ is homogeneous of order 1 in $|c_1|$ and $|c_2|$, a result which also follows from physical arguments. In particular

$$f(c_1, c_2, k\Delta) = (c_1^2 + c_2^2)^{\frac{1}{2}} f\left(\frac{c_1}{(c_1^2 + c_2^2)^{\frac{1}{2}}}, \frac{c_2}{(c_1^2 + c_2^2)^{\frac{1}{2}}}, k\Delta\right) \tag{32}$$

$$= (c_1^2 + c_2^2)^{\frac{1}{2}} f(\cos\theta, \sin\theta, k\Delta) \tag{33}$$

where

$$\theta = \arctan\frac{c_2}{c_1} \qquad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}. \tag{34}$$

Accordingly each of these functions (for $k = 1, 2, \ldots, r$) can be stored as a function of a single variable, and in terms of its values at points on the unit circle $c_1^2 + c_2^2 = 1$. Since Ω is a function of the moduli of u and u'', $f(c_1, c_2, k\Delta) = f(-c_1, -c_2, k\Delta)$, and so the function need only be defined on the unit semicircle

$$c_1^2 + c_2^2 = 1$$
$$c_1 > 0.$$

The interval $-(\pi/2) \le \theta \le (\pi/2)$ is normalized to $[-1, 1]$, by introducing a new variable $x = 2\theta/\pi$, and the functions have been represented by truncated series of orthogonal polynomials, thus

$$f(\cos\theta, \sin\theta, k\Delta) = \sum_{i=0}^{n} a_{ki}\phi_i(x) \tag{35}$$

where ϕ_i is the Legendre polynomial of order i [23]. Once $f(\cos\theta, \sin\theta, k\Delta)$ has been found, using (30) and $f(\cos\theta, \sin\theta, (k-1)\Delta)$ represented by (35),

$$a_{ki} = \int_{-1}^{1} f\left(\cos\frac{\pi x}{2}, \sin\frac{\pi x}{2}, k\Delta\right)\phi_i(x)\,dx. \tag{36}$$

How can this integration best be carried out numerically? It might be done by Simpson's rule, which requires tabulation of f at regular intervals $-1, -1+\delta, \ldots, +1$; in that case, however, it would clearly be equally accurate simply to store and use tables of $f(\cos(\pi x/2), \sin(\pi x/2), k\Delta)$ and find intermediate values by interpolation. It is much better to use a more sophisticated numerical integration technique, and write

$$a_{ki} = \sum_{j=1}^{m} f\left(\cos\frac{\pi x_j}{2}, \sin\frac{\pi x_j}{2}, k\Delta\right)\phi_i(x_j)w(x_j) \tag{37}$$

where the x_j are the abscissae for m-point Gaussian integration and $w(x_j)$ the corresponding weight functions. At each Δ-step in the ξ-direction the minimum in (30) need then only be determined at m points. Since the f functions are reasonably smooth, accurate representation of the functions can be achieved with relatively few points.

The minimization in (30) cannot be done by calculus, since Ω is not continuously differentiable. Instead the minimum is found by direct comparison of values, using the "golden section" modification of the optimal Fibonnacci search for extreme values of unimodal functions (Kiefer, [24]). Golden section search is described by Wilde [25]; it is almost as efficient as Fibonnacci search and easier to programme. In a search for the minimum value of a unimodal function, which is initially known to lie in a certain interval, s evaluations of the function are enough to reduce the interval within which the minimum must lie in the ratio $(\frac{1}{2}(\sqrt{5}-1))^{(s-1)}$, which is $<0{\cdot}01$ for $s = 11$ and $<0{\cdot}001$ for $s = 16$.

If this method is to work the function to be minimized must be a unimodal function of a single variable. In this case the function within the square brackets of (30) is to be minimized for fixed c_1 and c_2: it is a function of u''. From its definition (24), $\Omega(u, u'')$ is convex. From arguments ultimately deriving from stability, it is shown in an Appendix to this paper that $f(u, u', \lambda/2)$ is for any λ a convex function of u and u'. It follows that

$$\Omega(u, u'')\Delta + f(u + v\Delta, v + u''\Delta, k\Delta)$$

is the sum of two convex functions of u'', and therefore itself convex. Since it is convex it must be unimodal.

At each step in ξ, (30) is used m times to determine f at each of the m integration points. Once this has been done (37) determines the n coefficients a_{ki}, and (33) and (35) can be combined to determine $f(u, u', k\Delta)$ for any u and u'; only the a_{ki} have to be stored.

From (25–27), the limit ring load for a centrally loaded cylinder of half-length $\lambda/2$ is $2f(1, 0, \lambda/2)$, which is given at once by the polynomial representation. The associated velocity field which minimizes the upper bound is determined by retracing steps in the axial ξ-direction. Consider a shell of half-length $\lambda/2 = r\Delta$; the curvature u'' over the length interval $(0, \Delta)$ minimizes

$$\Omega(1, u'')\Delta + f(1, u''\Delta, (r-1)\Delta)$$

starting from the initial conditions $u(0) = 1$, $u'(0) = 0$. This minimization is carried out in the same way as the others, using the stored coefficients a_{ki}. If the minimizing value of u'' is z, the associated velocity field in the interval $(0, \Delta)$ is

$$u = 1 + \frac{z\xi^2}{2} \tag{38}$$

and the initial conditions are transformed, into

$$\begin{aligned} u &= 1 + \frac{z\Delta^2}{2} \\ u' &= z\Delta. \end{aligned} \tag{39}$$

Applying the recurrence relation a second time, the curvature over the second interval $(\Delta, 2\Delta)$ minimizes

$$\Omega(1 + \tfrac{1}{2}z\Delta^2, u'')\Delta + f(1 + \tfrac{1}{2}z\Delta^2, z\Delta, (r-2)\Delta)$$

and is identical with the curvature for the *first* ξ–interval of a shell of half-length $(r-1)\Delta$ where u and u' have the initial values (39). Repeating this process constructs the complete velocity field.

The velocity field might alternatively have been constructed by storing decision functions giving the optimal curvature as a function of u, u' and k, in the same way as the f functions were stored, and then retracing steps from $(1, 0, r\Delta)$. Although this method appears attractive, it is unsuitable because unlike $f(u, u', k\Delta)$ the optimal $u''(u, u', k\Delta)$ is not a smooth function of its arguments, or even necessarily continuous; it cannot therefore easily be represented by polynomials.

NUMERICAL RESULTS

An autocode programme for the Cambridge University Titan computer was written to carry out the calculations described in the previous section. As was explained earlier, it was decided not to include hinge discontinuities in u' explicitly but to gain the same effect by allowing large but finite curvatures u''. There is no reason why Δ should have the same value for each step in the ξ direction, and it was found better to make alternate long steps of length d_1 and short steps of length d_2, where abrupt changes in slope will not produce large changes in u. In determining the appropriate values of u and u' in the expression to be minimized (equation (30)) second-order terms were included so that (30) became

$$f(c_1, c_2, k\Delta) = \min_{u''}[\Omega(c_1+\tfrac{1}{2}c_2\Delta+\tfrac{1}{6}u''\Delta^2, u'')\Delta+f(c_1+c_2\Delta+\tfrac{1}{2}u''\Delta^2, c_2+u''\Delta, (k-1)\Delta)].$$

Ten-point Gaussian integration combined with Legendre polynomials up to order 9 gave collapse loads within 2 per cent of the analytic values, but less satisfactory edge load interaction diagrams (see below). Experiment showed that better results were given by 20-point integration, and all the results described in detail here were obtained with

$d_1 = 0{\cdot}09$	long step in axial direction
$d_2 = 0{\cdot}01$	short step in axial direction
$m = 20$	number of integration points
$n = 9$	order of highest order Legendre polynomial
$s = 10$	number of comparisons in minimization search
	and limits ± 100 on u'', allowing a change of slope of 1 in a short step 0·01.

The computation time required for 50 steps in the ξ-direction was approximately 3 min. In this time the coefficients a_{ki} and the collapse loads for centrally loaded shells are determined—once this is done very little extra time is required for the solution of any collapse analysis problem for a cylindrical shell under a single ring load.

In Fig. 3 collapse ring load upper bounds for a centrally loaded shell are compared with exact values from Demir's analysis [18]. Agreement is encouraging. Though the computation should give an upper bound, rounding errors and the smoothing effects of polynomial representation cause a few of the calculated points to lie very slightly below the calculated curve, and to increase very slowly in the region where the limit load should be constant and equal to that in an infinite shell.

The calculated modes minimizing the upper bounds for shells of non-dimensionalized half-lengths 0·5, 1·0, . . . , 2·5 are plotted in Fig. 4. A comparison with Eason's result for an

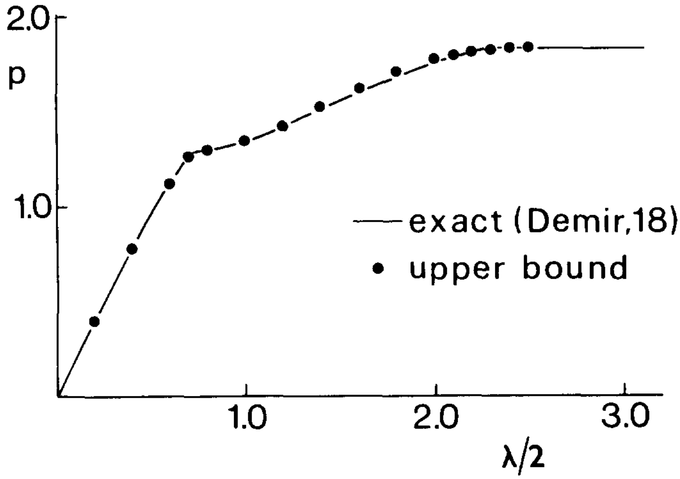

FIG. 3.

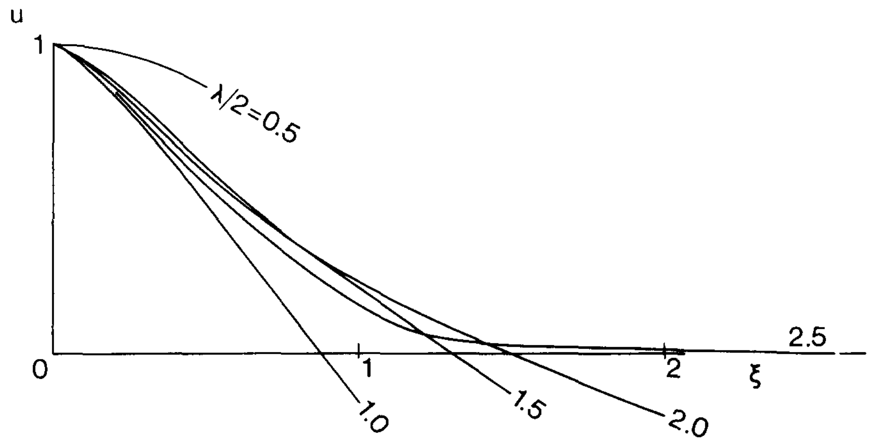

FIG. 4.

infinite shell [17] shows that the general form is given well, but that hinge circles tend to be slightly smoothed; this occurs even though the limits on u'' are wide enough not to constrain the value of u'' chosen by the minimization process. This appears to be explained by the fact that if u is small compared to u'' the dissipation function Ω is insensitive to small changes in u''.

Consider a cylindrical shell of length $L/2$ loaded at one end by radial forces of intensity Q/unit circumference and by a ring moment of intensity M (Fig. 5). If $U(x)$ is any velocity field in $0 < x < L/2$ and Q^*, M^* are some combination of Q and M which produce collapse, then from the upper bound theorem

$$Q^*U(0)+M^*U_x(0) \le \int_0^{L/2} D\left(U, \frac{d^2U}{dx^2}\right)dx. \tag{40}$$

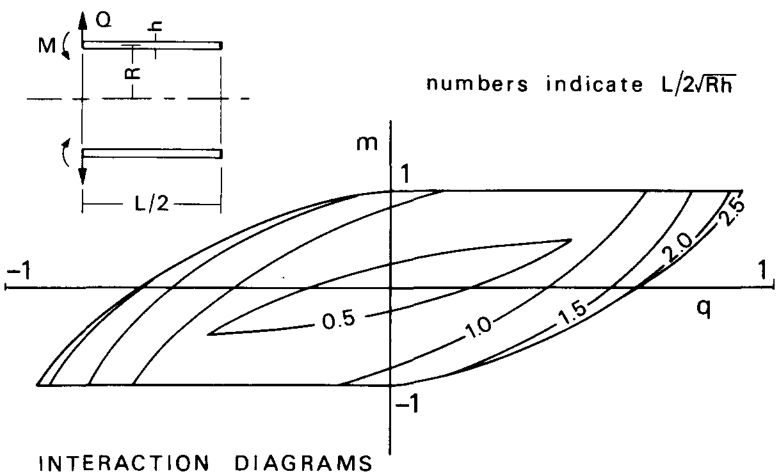

FIG. 5. Interaction diagrams; numbers indicate $L/2\surd(Rh)$.

Non-dimensionalizing by (19), with

$$q = \frac{\surd(Rh)}{\sigma_0 h^2} Q \tag{41}$$

this becomes

$$q^* u(0) + \tfrac{1}{4} m^* u'(0) \leq \int_0^{\lambda/2} \Omega(u, u'')\, d\xi. \tag{42}$$

Since the minimum value of the integral is $f(u(0), u'(0), \lambda/2)$,

$$q^* u(0) + \tfrac{1}{4} m^* u'(0) = f(u(0), u'(0), \lambda/2) \tag{43}$$

$$q^* \cos\theta + \tfrac{1}{4} m^* \sin\theta = f(\cos\theta, \sin\theta, \lambda/2) \tag{44}$$

$$= \sum_{i=0}^{n} a_{ri} \phi_i(2\theta/\pi) \tag{45}$$

where $\lambda/2 = r\Delta$. This represents a family of lines in Q^*, M^* space with parameter θ. Its envelope is the interaction diagram representing the combinations of m^* and q^* which will produce collapse in the shell, and once the a_{ri} have been determined the diagram can easily be constructed. The diagrams for $\lambda/2 = 0{\cdot}5, 1{\cdot}0, \ldots, 2{\cdot}5$ are given in Fig. 5. Calladine [20] has constructed the corresponding diagrams for a simpler shell yield locus. At each point on the locus the $u(0), \frac{1}{4}u''(0)$ vector is normal to it; the "cut-off" at $m = 1$ corresponds to rotation without radial movement in which a plastic hinge forms at $x = 0$.

CONCLUSIONS

In this example at least, the limit loads given by upper bound minimization through dynamic programming are extremely close to exact values from analysis. One cannot be certain that this will always be the case, and it may be useful to complement the method by a lower bound analysis by linear programming or the SUMT technique. An attractive feature of the method is that a large class of "neighbouring" problems is solved at the same time; this would be useful in design applications. The method can be extended to rotationally symmetric shells of more complicated shape, where a single variable is no longer sufficient to describe the deformation, to more complex yield conditions, and to direct design problems. Computational difficulties may arise if the minimization in the functional equation is over a large number of variables, but such minimization problems have been intensively studied, and a number of sophisticated methods for their solution exist.

One question remains: is a velocity field which minimizes an upper bound necessarily identical with the velocity field of the complete solution? In general it will not be identical; simple examples show that upper bound loads coincident with exact solution loads may be given by velocity fields very far from complete solution velocity fields. Close to exact solutions load upper bounds may be very insensitive to details of velocity fields. Intuitively, however, one expects velocity fields that minimize upper bounds to be "close" to exact solution except in very unconstrained problems, and this appears to be supported by the cylinder problem considered here.

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APPENDIX

$f(u, u', \lambda/2)$ is a convex function of u and u'

The function $f(u, u', \lambda/2)$ represents the dissipation in a yielding shell of length $\lambda/2$, which has imposed radial and angular velocities u and u' at one end, the other end being free. Hypersurfaces of constant rate of energy dissipation have been studied by Calladine and Drucker [26]. In displacement space—in this case in u, u' space—these surfaces are convex, enclose the origin, and nest so that each one lies outside all surfaces corresponding to smaller dissipation rates. Consider for simplicity the case of a general structure with two displacement components u and v. The associated dissipation rate is $f(u, v)$, homogeneous of order 1 in u and v.

Let (u_1, v_1) and (u_2, v_2) be distinct points in displacement space, labelled A and B in Fig. 6; suppose $f(u_2, v_2) > f(u_1, v_1)$. The contour $f = f(u_1, v_1)$ intersects OB at a

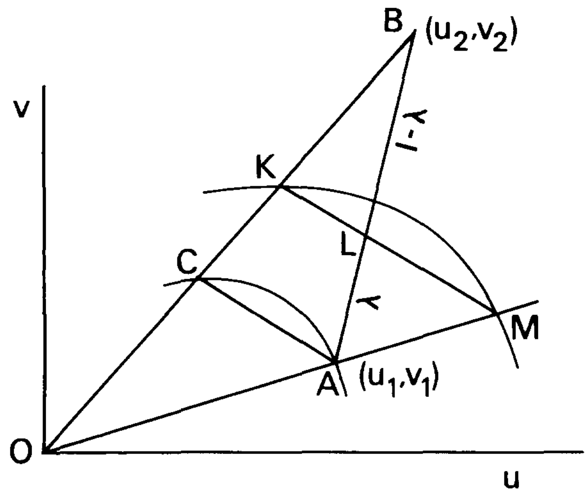

FIG. 6.

point C between O and B. Denote by L the point $(\lambda u_1 + (1-\lambda)u_2, \lambda v_1 + (1-\lambda)v_2)$ which divides AB in the ratio $\lambda : 1-\lambda$ $(0 < \lambda < 1)$. A line through L parallel to AC intersects OA produced at M and CB at K; it divides CB in the ratio $\lambda : 1-\lambda$ (from geometry). Since the function f is homogeneous the contour through K also passes through M, and since contours are convex, the point L must line on or inside this contour. But at K f has the value

$$\lambda f(u_1, v_1) + (1-\lambda) f(u_2, v_2)$$

since f is homogeneous and has the same value at C as at A. It follows that

$$f(\lambda u_1 + (1-\lambda)u_2, \lambda v_1 + (1-\lambda)v_2) \leq \lambda f(u_1, v_1) + (1-\lambda) f(u_2, v_2)$$

for any λ in (0, 1), the condition that f be a convex function.

(*Received* 7 *June* 1968; *revised* 23 *September* 1968)

Абстракт—Путем определения наименьшей верхней границы нагрузки разрушения ротационной, симметрической оболочки, задачу можно свести к исследованию минимума интеграла, подверженного некоторым условиям. Динамическое программирование дает способ доведения этого исследования минимума, не ограниченного отсуствием гладкости интегрирующей функции. Указанный способ приводит к обыкновенному расчетному алгоритму. В работе разработывается метод расчета и применяется его к тонкой цилиндрической оболочке, подверженной кольцевой нагрузке. Результаты близко сходятся с точным способом расчета.